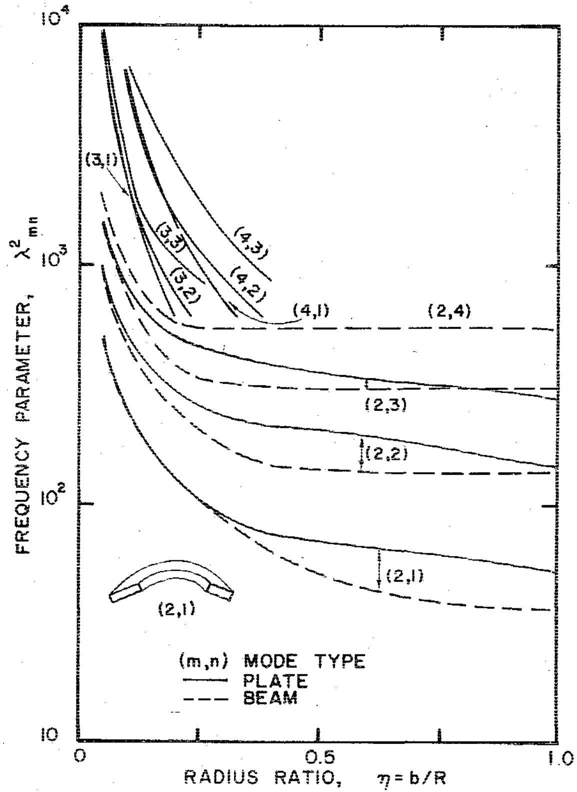

Fig. 3 Dominant twisting frequencies for $\theta_0 = 30$ deg, $\nu = 0.3$.

frequencies p [Eq. (11)] for a straight beam of the same length and stiffness as the respective curved segment. Corresponding curved element frequencies show increasing depression from these reference values as both η and θ_0 increase. Second, the fundamental frequency for plates is always lower than for beams, other parameters remaining the same. Third, although frequencies for $m \geq 3$ for $n = 1,2,...$ may be significant for $\eta > 0.2$, such frequencies are very high by comparison to those for $m = 1,2$ and $\eta > 0.2$. This implies that, for the low range of η, the curved beam theory can be employed with reasonable confidence in the dynamic design of curved elements. This is because there is little difference between the natural vibration frequencies calculated from plate theory and curved beam theory for ratios $\eta = b/R < 0.2$.

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Condensation of Free Body Mass Matrices Using Flexibility Coefficients

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REDUCTION of mass matrices for vibration analysis has become a routine part of many structural analysis systems. NASTRAN[1] includes a procedure by Guyan.[2] Kaufman and Hall[3] and Ramsden and Stoker[4] have suggested a procedure using flexibility coefficients. The procedure by Kaufman and Hall and Ramsden and Stoker is, however, limited to a restrained structure such that the flexibility matrix E is equal to the inverse of the stiffness matrix K.

The Guyan reduction is based on partitioning the stiffness matrix according to the deflection subsets U_s and U_o such that

$$\begin{bmatrix} K_{ss} K_{so} \\ K_{os} K_{oo} \end{bmatrix} \begin{Bmatrix} U_s \\ U_o \end{Bmatrix} = \begin{Bmatrix} P_s \\ O \end{Bmatrix} \tag{1}$$

where U_s are the retained d.o.f. (degrees of freedom) and U_o are the omitted d.o.f. This leads to a coordinate transformation of the form

$$\begin{Bmatrix} U_s \\ U_o \end{Bmatrix} = \begin{bmatrix} I \\ -K_{oo}^{-1} K_{os} \end{bmatrix} \{U_s\} \tag{2}$$

The method suggested by Kaufman and Ramsden and Stoker can be shown to be equivalent, if the flexibility matrix E is equal to K^{-1} such that

$$\begin{bmatrix} K_{ss} K_{so} \\ K_{os} K_{oo} \end{bmatrix} \begin{bmatrix} E_{ss} E_{so} \\ E_{os} E_{oo} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \tag{3}$$

Expansion to obtain the off-diagonal terms leads to

$$K_{os}E_{ss} + K_{oo}E_{os} = 0$$

Therefore

$$-K_{oo}^{-1} K_{os} = E_{os}E_{ss}^{-1}$$

The resulting transformation is

$$\begin{Bmatrix} U_s \\ U_o \end{Bmatrix} = \begin{bmatrix} I \\ E_{os} E_{ss}^{-1} \end{bmatrix} \{U_s\} \tag{4}$$

For a free body the stiffness matrix is singular and K^{-1} does not exist. In this case we must define a subset of rigid body d.o.f. U_r and partition K such that

$$\left[\begin{array}{cc|c} K_{ss} & K_{sr} & K_{so} \\ K_{rs} & K_{rr} & K_{ro} \\ \hline K_{os} & K_{or} & K_{oo} \end{array}\right] \begin{Bmatrix} U_s \\ U_r \\ \hline U_o \end{Bmatrix} = \begin{Bmatrix} P_s \\ P_r \\ \hline O \end{Bmatrix} \tag{5}$$

The reduction transformation is

$$\begin{Bmatrix} U_s \\ U_r \\ U_o \end{Bmatrix} = \left[\begin{array}{c|c} I & O \\ O & I \\ -K_{oo}^{-1} K_{os} & -K_{oo}^{-1} K_{or} \end{array}\right] \begin{Bmatrix} U_s \\ U_r \end{Bmatrix} \tag{6}$$

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To obtain an equivalent expression using the flexibility coefficients from Eq. (3), we write

$$\begin{Bmatrix} U_s \\ U_o \end{Bmatrix} = \begin{bmatrix} E_{ss} & E_{so} \\ E_{os} & E_{oo} \end{bmatrix} \begin{Bmatrix} P_s \\ O \end{Bmatrix} + \begin{bmatrix} \Phi_{sr} \\ \Phi_{or} \end{bmatrix} \{U_r\} \qquad (7)$$

where Φ_{sr} and Φ_{or} are the rigid body mode shapes for the s d.o.f. and the o d.o.f., respectively.

Expand Eq. (7) to obtain

$$U_s = E_{ss}P_s + \Phi_{sr}U_r \qquad (7a)$$

$$U_o = E_{os}P_s + \Phi_{or}U_r \qquad (7b)$$

Multiply Eq. (7a) by E_{ss}^{-1} and solve for P_s; then substitute into Eq. (7b).

$$U_o = E_{os}E_{ss}^{-1}U_s + (\Phi_{or} - E_{os}E_{ss}^{-1}\Phi_{sr})U_r$$

The transformation, compatible with Eq. (6), is

$$\begin{Bmatrix} U_s \\ U_r \\ U_o \end{Bmatrix} = \begin{bmatrix} I & O \\ O & I \\ E_{os}E_{ss}^{-1} & (\Phi_{or} - E_{os}E_{ss}^{-1}\Phi_{sr}) \end{bmatrix} \begin{Bmatrix} U_s \\ U_r \end{Bmatrix} \qquad (8)$$

The transformation of Eq. (8) can be used to reduce the mass matrix. This method requires only the rigid body modes in addition to the deflection matrices, E_{os} and E_{ss}, and the inverse matrix E_{ss}^{-1}. The "s" set is much smaller than the "o" set, therefore, this method should be faster and less expensive than the Guyan reduction, since it is not necessary to invert K_{oo}.

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Design Plastic Stress Concentration Factors Using Ramberg-Osgood Stress-Strain Parameters

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Nomenclature

- E = modulus of elasticity
- K_t = theoretical elastic stress or strain concentration factor
- K^* = limit value of K_t for elastic reference stress
- K_σ = plastic stress concentration factor
- K_ϵ = plastic strain concentration factor
- m = Ramberg-Osgood exponent
- R = plastic stress concentration reduction factor
- R^* = reduction factor for discontinuity stress in excess of secant-yield
- α = end-strain ratio
- ϵ = strain
- ϵ_n = discontinuity strain
- ϵ_0 = reference strain
- ϵ_I = secant-yield strain
- ϵ_α = end-strain of Ramberg-Osgood equation
- σ = stress
- σ_n = discontinuity stress
- σ_p = effective proportional limit stress
- σ_0 = reference stress
- σ_I = secant-yield stress
- σ_α = end-stress of Ramberg-Osgood equation

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Background

SPECIFIC methods for deriving plastic concentration factors were recently described using graphic[1] and analytic[2] methods. In each case the Neuber plastic concentration factor equation[3] was used in conjunction with an analytical approximation of the stress-strain curve. In Ref. 2 results were developed using a two-piece approximation of the stress-strain curve: the Ramberg-Osgood equation[4] up to the secant-yield stress and a simple power law thereafter.[5] The rationale for the piecewise formulation was that the curve-fitting procedure for the Ramberg-Osgood equation required experimental data only up to the secant-yield stress; hence there was no assurance that the equation was suitable beyond this limit. The possibility exits, however, that a statistically satisfactory fit of experimental stress-strain data to the Ramberg-Osgood equation can be achieved for values in excess of the secant-yield stress for some materials; hence, a piecewise approximation with its added complexity may not be necessary.

It is the purpose of this Note to extend the analysis of Ref. 2, using only the Ramberg-Osgood equation, by developing relations between reference and discontinuity stresses in a form that may be useful in design, and from which nondimensional design graphs may be constructed. Included in the development is a method of specifying an end-point value for the Ramberg-Osgood equation when a satisfactory fit of experimental data can be made for stress and strain values beyond the secant-yield stress.

Ramberg-Osgood Equation

Expressed in stress-ratio form, the Ramberg-Osgood equation is:

$$(\sigma/\sigma_I) + (3/7)(\sigma/\sigma_I)^m = \epsilon/(\sigma_I/E) \qquad (1)$$

Note that the quantity on the right-hand side of Eq. (1) is the ratio of a total strain value to the elastic component of the secant-yield strain. This ratio has the value 10/7 when $\sigma = \sigma_I$. When the test data indicate a fit beyond σ_I a constant α can be specified that is related to the end-point strain ϵ_α by

$$\epsilon_\alpha/(\sigma_I/E) = (10/7)\alpha \qquad (2)$$

The implication of Eq. (2) is that the total strain ϵ_α for a stress σ_α is α times the total strain associated with the secant-yield stress. The end-stress value σ_α can be found when α is specified by a numerical solution of Eq. (1). This equation has the form

$$X^a + (3/7)X^b - A = 0 \qquad (3)$$

where $X = \sigma_\alpha/\sigma_I$, $A = (10/7)\alpha$, $a = 1$, and $b = m$. As previously indicated[2] a solution of Eq. (3) is easily accomplished with a desk calculator or by machine computation using a simple root-finder routine.